# MEASURES OF TECHNOLOGICAL CHANGE AND THE USE OF THE TRANSLOG FUNCTION

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## The Translog Function

The analysis of technological change has been carried out until recently mostly in terms of the ARROW, CHENERY, MINHAS and SOLOW production function (CES), or in terms of its limiting form, the COBB-DOUGLAS production function (CD). Such forms, however, place unacceptable restrictions on their parameters as maintained hypotheses. As stated by FUSS et al [(1)] "one should not attempt to test a hypothesis in the presence of maintained hypotheses that have less commonly accepted validity" as such a practice would transform the test on some working hypothesis into a test of the model itself, i.e., into a test of the maintained hypotheses. One such overly restrictive maintained hypothesis concerns the elasticity of substitution when there are more than two factors of production.[(2)]

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(1) In FUSS et al (1978) p. 223.

(2) In the multiple-factor production function, there are a number of definitions of elasticity of substitution depending on a) the variables which are held constant and b) the number of variables involved in the operation. FUSS et al, in FUSS and MCFADDEN (1978) pp. 240-1 classified such different definitions in terms of one factor - one price elasticity of substitution, two factor - one price elasticity of substitution and two factor - two price elasticity of substitution. They are usually evaluated at constant output, constant cost, or constant quantities and prices of factors not involved in the operation, and could be interpreted as long-run or short-run concepts depending on whether quantities of other factors are or are not allowed to vary.
MCFADDEN (1963) defines the Direct Elasticity of Substitution (DES), a two factor - two price short-run elasticity of substitution as the "elasticity of the input ratio of factors i and j with respect to the marginal rate of substitution of these factors, taken along a fixed isoproduct curve with all remaining factor quantities fixed" (p.74). On page 76 he defines the Shadow Elasticity of Substitution (S.E.S.), as a two factor - two price long-run elasticity of substitution which can be found by applying the two factor formula to a

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BERNDT et al (1973b), UZAWA (1962), MCFADDEN (1973), RUSSELL (1975) and GOLDMAN et al (1964), among others, have derived important results relating the issues of separability,

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pair of factor holding constant the prices of the remaining factors and the total cost (MCFADDEN expressed it in terms of a cost function). He shows that production functions with constant elasticities of substitution (defined there as Direct and Shadow Elasticities of Substitution), such as C.E.S. and C.D., will have to satisfy the following conditions globally:

a) there exists a partition $\{N_1, \ldots, N_s\}$ of the set $\{1, \ldots, n\}$ of factors such that

$$\sigma_{ij} \equiv \begin{cases} 1 \text{ for } i, j \in N_s,\ i \neq j \\ \sigma \text{ for } i \in N_r,\ j \in N_s,\ r \neq s \end{cases} \qquad (1)$$

where $\sigma$ is a positive finite constant, and

b) $i$ and $j$ in the same set imply $X_i F_i = X_j F_j$ for all positive bundles where $F_i = \frac{\partial F}{\partial X_i}$ and $F$ (.) is a production function.

These are obviously strong restrictions, although one would have difficulty in attempting to explain why it is any more so than those implied by a two factor C.E.S. form. Note that if the number of sets S in the partition equal the number of factors n, then all $\sigma_{ij}$ would be equal and constant.

BERNDT et al (1973b) discussed the ALLEN elasticity of substitution, defined as the response of derived demand for an input i to a change in price of input j, holding output and all other input prices constant. This could be classified as a <u>one input - one price long-run</u> concept, and it is the definition we shall be using with regard to the functional form used in this thesis:

$$\sigma_{ij} = \frac{\sum_{h=1}^{N} F_h X_h \, |\bar{F}_{ij}|}{X_i X_j \, |\bar{F}|} \qquad (2)$$

where $|\bar{F}|$ is the determinant of the bordered HESSIAN of first and second order partial derivatives of the production function F, and $|\bar{F}_{ij}|$ is the cofactor of $F_{ij}$ in $\bar{F}$. If the supply of inputs is perfectly elastic (i.e. prices are constant) and under competitive assumptions, then $\sigma_{ij} = \frac{E_{ij}}{\pi_j}$ where $E_{ij}$ is the cross price elasticity of derived demand

$$E_{ij} = \frac{\partial \ln X_i}{\partial \ln p_j} \quad \text{and} \quad \pi_j = \frac{P_j X_j}{\sum_{i=1}^{n} P_i X_i} \quad \text{is}$$

functional form and the elasticity of substitution.[3]

Suppose F ($\underline{X}$) is a function of a set of n inputs denoted by N = {1,...,n}. The set N can be partitioned in S subsets $\{N_1,\ldots,N_S\}$ where $N = N_1 \cup N_2 \cup \ldots \cup N_S$ and $N_r \cap N_t$ is empty. Separability is defined as the independence of the marginal rate of substitution between a pair of inputs from changes in the quantity of a third input.

So, separability implies, locally or globally, that

$$\frac{\partial\left(\frac{F_i}{F_j}\right)}{\partial \quad X_h} = 0, \quad \text{or} \quad F_j F_{ih} - F_i F_{jh} = 0.^{(4)} \qquad (3)$$

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Finally, UZAWA (1962) has shown that the AES, defined in terms of the cost function C ($\underline{p}$,Y) is equal to

$$\sigma_{ij} = \frac{C\ (\underline{p},Y)\ \dfrac{\partial^2 C\ (\underline{p},Y)}{\partial p_i \ \partial p_j}}{\dfrac{\partial C\ (\underline{p},Y)}{\partial p_i} \cdot \dfrac{\partial C\ (\underline{p},Y)}{\partial p_j}} = \frac{C\ (\underline{p},Y)\ \dfrac{\partial X_i\ (\underline{p},Y)}{\partial p_j}}{X_i\ (\underline{p},Y)\ .\ X_j\ (\underline{p},Y)} = \frac{C\ (\underline{p})\ \dfrac{\partial^2 C\ (\underline{p})}{\partial p_i \ \partial p_j}}{\dfrac{\partial C\ (\underline{p})}{\partial p_i} \cdot \dfrac{\partial C\ (\underline{p})}{\partial p_j}},$$

using SHEPHARD's Lemma and assuming constant returns to scale (C ($\underline{p}$,Y) = Y C ($\underline{p}$)). BINSWANGER (1974a) showed this result to be true without the homogeneity assumption.

(3) The issue of functional separability is directly related to the issue of aggregation and derivation of value added functions, and is one of the main points of interest in functional forms, such as the translog production function which do not assume separability as a maintained hypothesis.

(4) Mutatis mutandis, the same definition can be applied to the cost function C ($\underline{p}$,Y). Actually LAU, in FUSS et al (1978) shows that separability properties in cost functions imply identical properties in the production functions from which they are derived, i.e. separability properties in input prices imply identical conditions in input quantities, given a homothetic production function and its dual cost function.

A function is weakly separable if (3) holds for all $i, j, \in N_r$ and $h \notin N_r$; a function is strongly separable if (3) holds for $i \in N_r$, $j \in N_t$ and $h \notin N_r \cup N_t$; complete separability implies a partition such that $S = n$, that is, each input forms its own subset.

GOLDMAN et al (1964) showed that functions of the form $F(X^1, X^2, \ldots, X^S)$ and $F(X^1 + X^2 + \ldots + X^S)$ where $X^r$ is a function of the elements of subset r only, imply weak and strong functional separability, respectively. Furthermore, BERNDT et al (1973b) have showed that there is a relationship between separability and the ALLEN elasticity of substitution, namely that the imposition of separability restriction on quasi-concave homothetic production function $F(\underline{X})$ is equivalent to certain equality and constancy restrictions on its ALLEN elasticities of substitution. Locally, weak separability (strong separability) is a necessary and sufficient condition for all $\sigma_{ih}$, $\sigma_{jh}$ to be equal, $i, j \in N_s$, $h \notin N_s$ ($i \in N_s$, $j \in N_r$, $h \notin N_s \cup N_r$); also, complete separability is a necessary and sufficient condition for all $\sigma_{ij}$ to be equal.

Global separability implies both equality and constancy of the ALLEN elasticities of substitution. Complete separability and equality and constancy of all $\sigma_{ij}$ are equivalent restrictions on $F(\underline{X})$. Furthermore, the multi-factor C.E.S. function is the only form compatible with global complete separability within the class of quasi-concave homothetic functions.(5) Thus, we

(5) The results obtained by MCFADDEN (1973) described in footnote 2 were not derived for the ALLEN elasticity of substitution. However, as stated by BERNDT et al (1973b), the ALLEN, Direct and Shadow elasticities of substitution coincide when F has constant returns to scale and all ALLEN elasticities of substitution are equal. This last condition holds if there is complete separability. Since the ARROW, CHENERY, MINHAS and SOLOW function is completely separable, conditions (3.1) also hold for the ALLEN elasticity of substitution in this particular case, and is consistent with BERNDT et al (1973b) results, specifically theorems 5 and 6.

see that both the C.E.S. and the C.D. form impose unacceptable restrictions on the substitutability parameter of the production function, namely requiring them to be all equal and constant. Ideally, one would like to have flexible functional forms which imposed as few restrictions as possible as maintained hypotheses, (6) linear in parameters for ease of estimation and with minimal number of parameters.

Furthermore, instead of using production functions to estimate technology parameters, recent developments in the duality theory allow the recuperation of technological parameters of production functions through estimation of its dual, the cost function. In what follows, we shall first analyse the generation and main properties of functional forms with the characteristics described above, then the main results of the duality theory will be presented, and finally the functional form to be used in the empirical estimation of this thesis will be presented and its main characteristics discussed.

A common method of generating general, flexible, minimal and linear in parameters functional forms is the use of TAYLOR Series second-order expansions to approximate arbitrary functions. (7) In general arbitrary functions F ($\underline{X}$) can be represented by a TAYLOR Series second-order expansion around point $\underline{a}$ as

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(6) Specifically with respect to the elasticity of substitution it is reasonable to ask for a functional form that allows for arbitrary values between any two pairs of inputs.

(7) FUSS et al, in FUSS and MCFADDEN (1978), discuss the use of the TAYLOR series in regard to such forms as the C.E.S., C.D., Generalized LEONTIEFF, Translog, Generalized C.D., Quadratic and Generalized Concave functions. Such functions are approximations to an arbitrary function about a point $X^*$, and therefore its accuracy will be acceptable in a small neighbourhood of that point. In other regions of interest, the approximation may be quite poor and may even fail convexity and monotonicity properties.

$$F\left(a_1 + X_1 + a_2 + X_2, \ldots, a_n + X_n\right) = F\left(a_1, a_2, \ldots, a_n\right) + \\ + \left(X_1F_1 + X_2F_2 + \ldots + X_nF_n\right)_{\underline{a}} + \frac{1}{2}\left(X_1{}^2F_{11} + X_2{}^2F_{22} + \ldots + X_n{}^2F_{nn} + \\ + 2\, X_1X_2F_{12} + 2\, X_1X_3F_{13} + \ldots + 2\, X_{n-1}X_nF_{n-1,n}\right)_{\underline{a}} + \text{Remainder} \quad (5)$$

where the subscript a indicates that all partial derivatives are to be evaluated at that point.[8] A second-order approximation to the function F ($\underline{X}$) would equal ( 5) without the remainder.

CHRISTENSEN et alii (1973) introduced the transcendental logarithm production function (translog). It can be interpreted as an approximation to an arbitrary function Y = F ($\underline{X}$) by a second-order TAYLOR Series expansion of ln Y on powers of ln $X_i$. So,

$$\ell n\, \underline{Y} = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ell n X_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} \ell n\ X_i \ell n X_j \qquad (6)$$

where $\alpha_0$ equals the value of $\ell nF(\underline{a})$, $\alpha_i$ are the first partial derivatives of $\ell nF$ with respect to $i^{th}$ variable and $\alpha_{ij}$ are its second-order partial derivatives, evaluated at the point of expansion a. With $\alpha_0$ $\alpha_i$ and $\alpha_{ij}$ considered constants, function ( 6) is a production function in its own right, having the properties we described above as being desirable. It is a general functional form which does not impose, a priori,

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(8) Any function can be approximated by a TAYLOR series expansion up to any desired order, as long as the infinite series is convergent. This requires that the remainder converge to zero as n tends to infinity. The remainder is defined in ALLEN (1938) pp. 456; in more compact notation, equation (3.5) can be expressed as

$$F\ (\underline{a} + \underline{X}) = F(a) + \sum_{i=1}^{n} X_1F_1 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} X_iX_jF_{ij} + \text{Remainder} \qquad (7)$$

separability and homogeneity as maintained hypotheses, allowing therefore for arbitrary values of the elasticity of substitution between any pair of inputs; it is linear and has a minimal number of parameters. At the same time, restrictions on the values of its parameters can be imposed and, therefore, it can be used to test assumptions such as homogeneity, separability, technical change and other implications from the theory of production.

Production functions should satisfy certain regularity conditions to produce meaningful economic results.(9) Unfortunately, the translog production function fails to satisfy the monotonicity and concavity conditions over the entire positive orthant.(10) It may however, satisfy the conditions locally making it necessary to test for them in regions of interest.

BERNDT et al (1973a) have shown the usefulness and maleability of the translog production function in estimating the substitutability between equipment, structures and labour in U.S. manufacturing for the period 1929-68.(11) They start with a translog production function in n inputs and in an index of technological change $\ell n\ V\ (X_1, X_2\ \ldots\ X_n,\ A)$. If there are constant returns to scale,

$$\ell n\ V\ (\lambda X_1, \lambda X_2, \ldots \lambda X_n, A) = \ell n\ V\ (X_1, X_2\ \ldots X_n, A) + \ell n \lambda$$

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(9) These conditions are the "Conditions I on Production Functions" described later in this chapter.

(10) All regularity conditions are satisfied globally only if $\alpha_i \geqslant 0$ for all i, $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_{ij} = 0$ for all i,j, in which case the translog function collapses into a COBB-DOUGLAS production function.

(11) Here we use their own notation.

If it is further assumed that there is HICKS neutral factor-augmenting technological change and constant return to scale, then

$$\ell n\ V\ (AX_1, AX_2, \ldots, AX_n) = \ell n\ V\ (X_1, X_2, \ldots, X_n) + \ell nA$$

where $\ell n\ V\ (X_1, X_2, \ldots, X_n)$ is called $\ell n$ F, so that $\ell n$ V can be expressed as $\ell n\ V = \ell n\ F + \ell n\ A$ or $V = FA$. Thus, individual inputs are transformed into an aggregate input $F\ (X_1, X_2, \ldots, X_n)$ by an input function F, and then such aggregate input is transformed into output by the technology index A.

Assuming that the input function F is translog, they showed that, with no loss of generality, they could estimate the parameters related to inputs $X_1 \ldots X_n$, limiting their attention to the F function, thus eliminating the technology index A from the estimating equations. Specifying that $F\ (X_1, X_2, X_3)$ is a translog constant return to scale input function and that inputs $X_1, X_2$ and $X_3$ (i.e. structure, equipment and labor) are weakly separable from the remaining inputs, they could write

$$V\ (X_1, X_2, X_3, \ldots, X_n, A) = V\left[F\ (X_1, X_2, X_3), X_4, \ldots, X_n, A\right]$$

using results obtained by GOLDMAN et al (1964) mentioned earlier. Thus, the authors were able to analyse the structure of production using a translog input function $F(X_1, X_2, X_3)$, abstaining from the technology index and from other inputs, but assuming F to be linear homogeneous. These are the inputs of interest for which elasticities of substitution are to be evaluated.

The translog function F is a general form. Assumptions such as constant returns to scale and neutrality of technological change can be imposed as restriction in the values of its parameters. Furthermore, it is interesting to observe that the translog form can be estimated in an unrestricted form and the assumptions which are made can be tested for. If accepted, the restricted translog form with the assumption included as a maintained hypothesis can be postulated.

BERNDT et al (1973a) proceed to derive all the desired separability hypotheses between inputs $X_1, X_2$ and $X_3$, in terms of required parametric restrictions, including the case of complete global separability, when the translog collapses into a COBB-DOUGLAS function. Finally the estimated parameters are tested for consistency with the postulated hypothesis.

The appeal of this study, in addition to the specific results about separability and, as a result, the existence or not of consistent aggregate input indices, is that it highlights the maleability and generality of the translog function, allowing for tests of assumed economic properties such as homogeneity, convexity, monotonicity, separability and types of technological change, which other commonly-used functional forms assume <u>a priori</u> to be true.

A similar demonstration of the qualities of the translog function can be obtained from CHRISTENSEN et alii (1973). DENNY et al (1977) used the translog function (production and cost functions) to test for the assumption of separability in Canadian manufacturing between capital and labour, on the one hand and intermediary inputs on the other. Separability is a necessary condition for the existence of real value added functions. They differ from BERNDT et al (1973a) methodology in that they interpreted the translog function as an approximation to a true function rather than as the exact one, thus relaxing somewhat the parametric constraints necessary for the existence of functional separability.

TODA (1976) used the C.E.S., the Generalized Linear and the Translog production functions to estimate elasticities of substitution for the Soviet Union. Instead of assuming HICKS neutrality as others have done, he introduced a labour-augmenting technological change specification in his estimated C.E.S., a factor-augmenting specification in his Generalized Linear and a general HICKS biased technological change specification in his estimates using the translog production function. We are not sure to what extent his estimates are comparable, given the different set of assumptions made for each of them.

CORBO et al (1979), using cross-section data from individual establishments in Chilean manufacturing estimated the unrestricted translog production function and tested for the suitability of the COBB-DOUGLAS form ( i.e. complete separability) and for the existence of economies of scale. For most industries they found evidence of the existence of constant returns to scale and of the suitability of the COBB-DOUGLAS form. -

BURGESS (1974) uses the translog form in analysing demand for inputs and followed BERNDT and CHRISTENSEN's approach for the implementation of the translog production function.

## Main Results of Duality Theory

Before we proceed to analyse the translog cost function, let us present the main results of the duality theory as far as it is relevant in connecting production and cost functions.

The duality theory could be defined as that part of economic theory which tries to analyse the implications and the extent to which certain properties attributable to basic concepts, such as a production function, carry over to economic constructs, such as a cost function and vice-versa. We know that given a production technology represented by a production function, it is possible to derive a cost function through the process of profit maximization behaviour, assuming competitive markets. The question we want to ask is the following: what can be known a priori about a cost function, given a production function with known properties? Alternatively, we would like to know what are the properties of a production function implied by a known cost function.

We shall see that there is an underlying duality between production functions and cost functions and that under certain conditions all the relevant information about technology can be recuperated from the study of cost functions without the need to derive the underlying production function. This result is of great importance, since it allows the study of technological change without the need to estimate a production function. It suffices to estimate the economic constructs based on the production function, such as cost functions, profit functions, factor demand functions or factor share functions. This procedure has innumerable advantages, among them the statistical convenience of greater availability and reliability of data and less stringent maintained hypotheses on the basic estimation model. Of course, we must assume competitive markets and profit maximization behaviour, which direct estimation of a production function must not. On the other hand, such assumptions are the basis for economic theory, and one way or another, they always creep in anyway in most studies based on the broad framework known as neoclassical theory.

There are two ways by which to derive systems of factor demand equations. The first method involves the solution of a constrained optimization problem using LAGRANGEAN or mathematical programming methods. Since the factor demand equations are solutions to such problems, it is possible to derive cost functions.(12)

Let's define the cost function as

$$C\ (\underline{p},Y) \equiv \min\ \underline{p}\ \underline{X}$$

$$\text{s.t. } Y = F(\underline{X}) \qquad (\ 8)$$

given a vector $\underline{p}$ of factor prices. Letting $X^*$ be the least expensive way to produce a certain level of output $\bar{Y}$, the first-order conditions to the optimization problem ( 8) yield the set of $\underline{n}$ factor demand equations,

$$X_i^*\ (\underline{p},Y) \qquad i = 1,\ldots,n \qquad (\ 9)$$

so that

$$C(\underline{p},Y) = \underline{p}.\underline{X}^* \qquad (\ 10)$$

This process of deriving factor demand equations and the cost function can be long, tedious and sometimes very complex. The production technology F ($\underline{X}$) has to be known, and, through cost minimizing behaviour in a competitive environment, it was possible to derive the cost function C($\underline{p}$,Y).

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(12) Constrained cost minimization problems yield conditional factor demands functions as solutions $X_i(\underline{p},Y)$, where the demand for factor i is a function of the vector of input prices given a determined output level Y. Constrained profit maximization problems yield unconditioned factor demand functions as solutions, $X_i\ (\underline{p},p_y)$ , where the demand for factor $\underline{i}$ is a function of the vector of input prices p and of output price $p_y$. In this study we shall be dealing with the former demand functions.

An alternative way to derive factor demand equations is simply to differentiate $C(\underline{p},Y)$ with respect to input prices. This result, known as SHEPHARD's lemma, allows us to differentiate $C(\underline{p},Y)$ and arrive at demand functions consistent with the optimizing behaviour described by (3.8). Thus,

$$X_i^*(\underline{p},Y) = \frac{\partial C(\underline{p},Y)}{\partial p_i} \qquad i = 1,\ldots,n. \qquad (\cdot 11)$$

This route certainly is much simpler than solving the constrained optimization problem ( 8), but its validity depends on assessing whether the properties of the production functions implied by the given cost function $C(\underline{p},Y)$ are acceptable, and on how they characterize the underlying production function $F(\underline{X})$. The duality theory does help in carrying out this task.

At the risk of being tedious, the second approach can be further exemplified by duality mapping in an attempt to draw conclusions about the properties of the underlying technology implied by a known cost function $C(\underline{p},Y)$. Such a cost function was derived from input requirement sets $V(Y)$ defined as

$$V(Y) = \left\{ \underline{X} : (Y, -\underline{X}) \in \mathcal{Y} \right\} \qquad ( 12)$$

and gives us all input bundles that produce at least $Y$,[13] so that

$$C(\underline{p},Y) = \underset{\underline{X}}{\text{minimum}} \left\{ p.\,\underline{X} : \underline{X} \in V(Y) \right\} \qquad ( 13)$$

(13) The vector $(Y, -\underline{X})$ represents a specific production plan with $Y$ as the output and the $\underline{X}$ as the vector of inputs. The negative sign indicates an input. $\mathcal{Y}$ is the set of all possible pattern of inputs and outputs, and gives a complete listing of technological possibilities available for production. It could alternatively be defined as

$$V(Y) = \left\{ \underline{X} : F(\underline{X}) \geqslant Y \right\} \qquad ( 14)$$

remembering that

$$F(\underline{X}) = \underset{Y \geqslant 0}{\text{maximum}} \left\{ Y : \underline{X} \in V(Y) \right\} \qquad ( 15)$$

Given the definition of a cost function, $V(Y)$ could also be defined as

$$V(Y) = \left\{ \underline{X} : p\underline{X} \geqslant C(Y,\underline{p}) \right\} \qquad ( 16)$$

Given $C(\underline{p},Y)$ what can we say about the production technology embodied in such a cost function? First it is necessary to construct such technology (duality mapping) and then use the duality theory to draw conclusions about it.

Figures 1 and 2 show us how to construct the technology implied by $C(\underline{p},Y)$.

DUALITY MAPPING

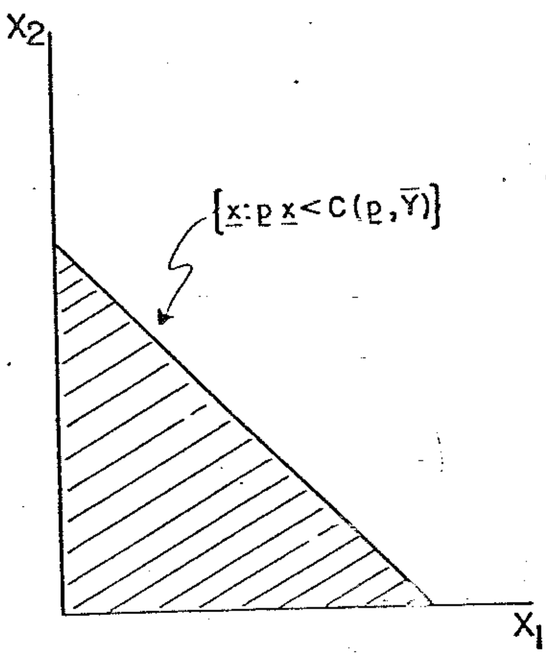


Figure 1

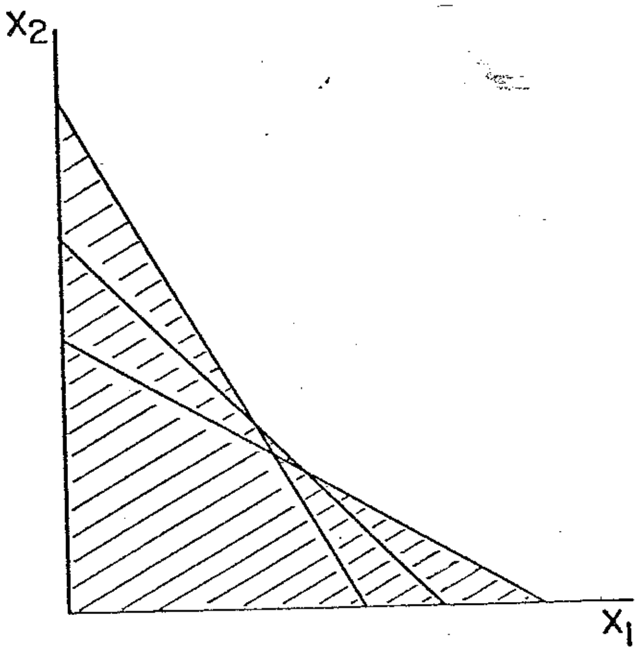


Figure 2

The technique involves the elimination of the $(X_1,X_2)$ points which do not belong to the input requirement sets ( 12). Using $C(\underline{p},Y)$ we draw isocosts for a given $\underline{p}$ and for a given level of output $\bar{Y}$. Figure 1 shows one such isocost. Any input bundle $\underline{X}$ in the shaded area cannot possibly produce $\bar{Y}$ since if it could, $\underline{pX} < C(\underline{p},Y)$ and by definition of the cost function, it would have been used.

In Figure 2 two more isocost lines are drawn, each related to a different set of input prices $\underline{p}$ and as a result we are able to eliminate further sub-sets of input bundles $\underline{X}$ which do not belong to $V(Y)$. Repeating this procedure for all possible $\underline{p}$, we derive input requirement sets $\tilde{V}(Y)$ described in Figure 3 by the shaded area.

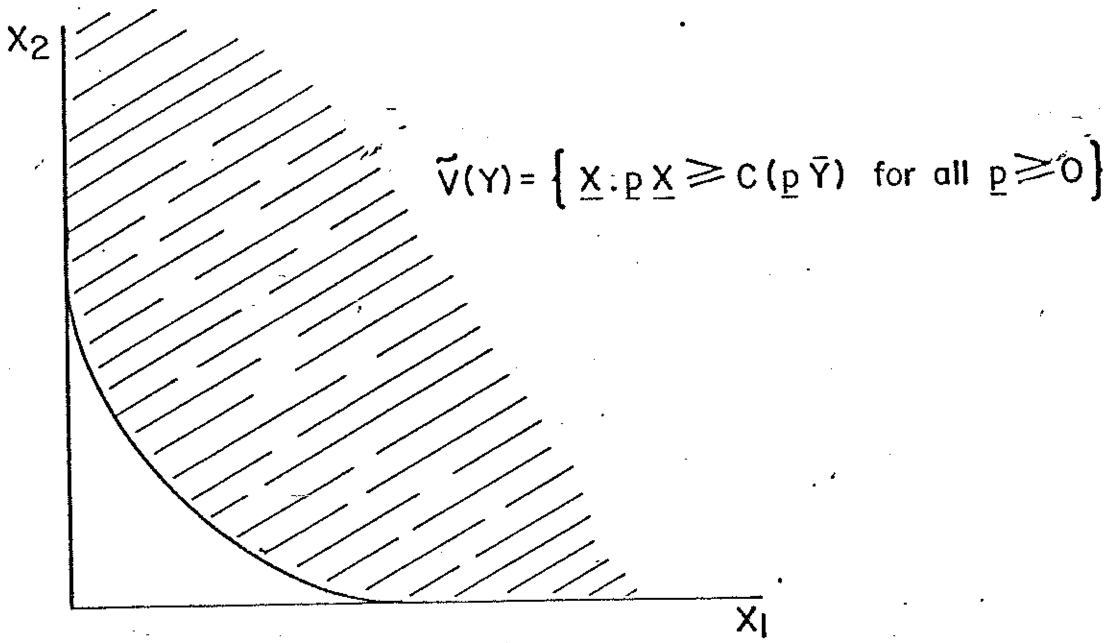


Figure 3 CONSTRUCTED INPUT REQUIREMENT SET $\tilde{V}(Y)$

It is possible to state that the yet unknown input requirement sets (the true technology) V(Y) are contained in $\tilde{V}(Y)$ since all input bundles which were not in $\tilde{V}(Y)$ were eliminated. We cannot say, however, that $V(Y) = \tilde{V}(Y)$ without examining the properties of those sets. This is what duality theory does, in trying to establish relationships between the "true" technology V(Y) and the constructed technology $\tilde{V}(Y)$ derived from a known cost function $C(\underline{p},Y)$.

The easier way of arriving at factor demand functions by applying SHEPHARD's Lemma can only make sense if there is a way of assuring that the relations derived from the cost function $C(\underline{p},Y)$ are consistent with an economically meaningful technology. If so, it is possible to state that all of the economically meaningful information carried by a production function is reflected and contained in its cost function and, therefore, it is possible to recuperate such information by simply analysing cost functions. This procedure has considerable advantages over the manipulation of production functions.

Following DIEWERT (1971), the main results of the SHEPHARD Duality Theorem are summarized, without proofs. These can be obtained directly from the cited source.(14)

We begin by stating three sets of conditions called "regularity conditions" which one would want to be satisfied by I) a production function, II) a family of input requirement sets and III) a cost function.

Conditions I on the Production Function F:

a) F is a real valued function defined for every input bundle $\underline{X} \geqslant 0$ and finite if each component of $\underline{X}$ is finite.

b) F is monotonic, that is, if $\underline{X}^1 \geqslant \underline{X}^2$ then $F(\underline{X}^1) \geqslant F(\underline{X}^2)$ (free disposal); also, $F(\underline{0}) = 0$.

c) $F(\underline{X})$ tends to infinity for at least one non-negative sequence of vectors (every output level is producible by some input bundle).

d) F is a right continuous function.

e) F is a quasi-concave function, that is, $V(Y) = \{ \underline{X} : F(\underline{X}) \geqslant Y \}$ is a convex set (diminishing returns).

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(14) In addition to DIEWERT (1971) the following sources were used in summarizing the main results of the duality theory: DIEWERT (1974), VARIAN (1978), UZAWA (1964), FUSS et alii in FUSS et al (1978), SHEPHARD (1953) and MCFADDEN (1966). DIEWERT (1974) has a brief historical note on the development of duality theorems where it can be seen that such results were partially known to many authors as early as 1932 and that they were frequently derived independently and/or simultaneously by different authors.

Conditions II on Input Requirements Sets V(Y):

a') $V(0)$ = the non-negative orthant; $V(Y)$ is non-empty, closed, and does not contain the origin for $Y > 0$.

b') $V(Y)$ is a convex set.

c') Monotonicity, that is, if $\underline{X}$ is in $V(Y)$ and $X^1 > \underline{X}$ then $\underline{X}^1$ also belongs to $V(Y)$.

d') If $Y^1 \geqslant Y^2$ then $V(Y^1)$ is a subset of $V(Y^2)$.

e') For every $\underline{X}$ there exists a Y such that $\underline{X}$ does not belong to $V(Y)$.

f') Graph V is a closed set.

Conditions III on Cost Function C(p,Y):

a") $C(\underline{p},Y)$ is positive, real valued and finite.

b") $C(\underline{p},Y)$ is a left continuous function in Y, and tends to infinity as Y tends to infinity.

c") $C(\underline{p},Y)$ is monotonic increasing in $\underline{p}$.

d") $C(\underline{p},Y)$ is linear, homogeneous in $\underline{p}$ for every Y, that is for $\lambda > 0$, $C(\lambda\underline{p},Y) = \lambda C(\underline{p},Y)$.

e") $C(\underline{p},Y)$ is a concave function in $\underline{p}$ for every Y that is for $0 \leqslant \lambda \leqslant 1$, $C(\lambda\underline{p}' + (1-\lambda)\ \underline{p}'',Y) \geqslant \lambda C(\underline{p}',Y) + (1-\lambda)\ C(\underline{p}'',Y)$ where $\underline{p}'$ and $\underline{p}''$ are two vectors of input prices.

We now reproduce four theorems, stated and proved by DIEWERT (1971), relating the production function F, the input requirement sets V(Y), and the cost function $C(\underline{p},Y)$.

Theorem 1 : A production function F satisfying conditions I implies that the input requirements sets V(Y) defined by ( 14) satisfy Conditions II.

Theorem 2: A family of input requirements sets V(Y) satisfying Conditions II implies that the production function F defined by ( 15) satisfy Conditions I. Moreover the input requirements sets which can be generated by F by means of ( 14) coincide with the original input requirement sets with which we started.

Theorem 3: A family of input requirements sets V(Y) satisfying Conditions II implies that the cost function $C(\underline{p},Y)$ defined by ( 13) satisfy Conditions III.

Theorem 4: A cost function $C(\underline{p},Y)$ satisfying Conditions III implies that the input requirement sets V(Y) defined by ( 16) satisfy Conditions II. Moreover the cost function $C(\underline{p},Y)$ which can be generated by V(Y) by means of ( 13) coincide with the original cost function with which we started.

These theorems are represented in Figure 4 to facilitate their understanding. For instance, the middle part of Figure 4, representing theorem 3, is simply telling us that a cost function satisfying Conditions III can be derived by use of definition ( 13) from a family of input requirement sets which satisfy Conditions II. The other parts of the Figure can be analogously interpreted.

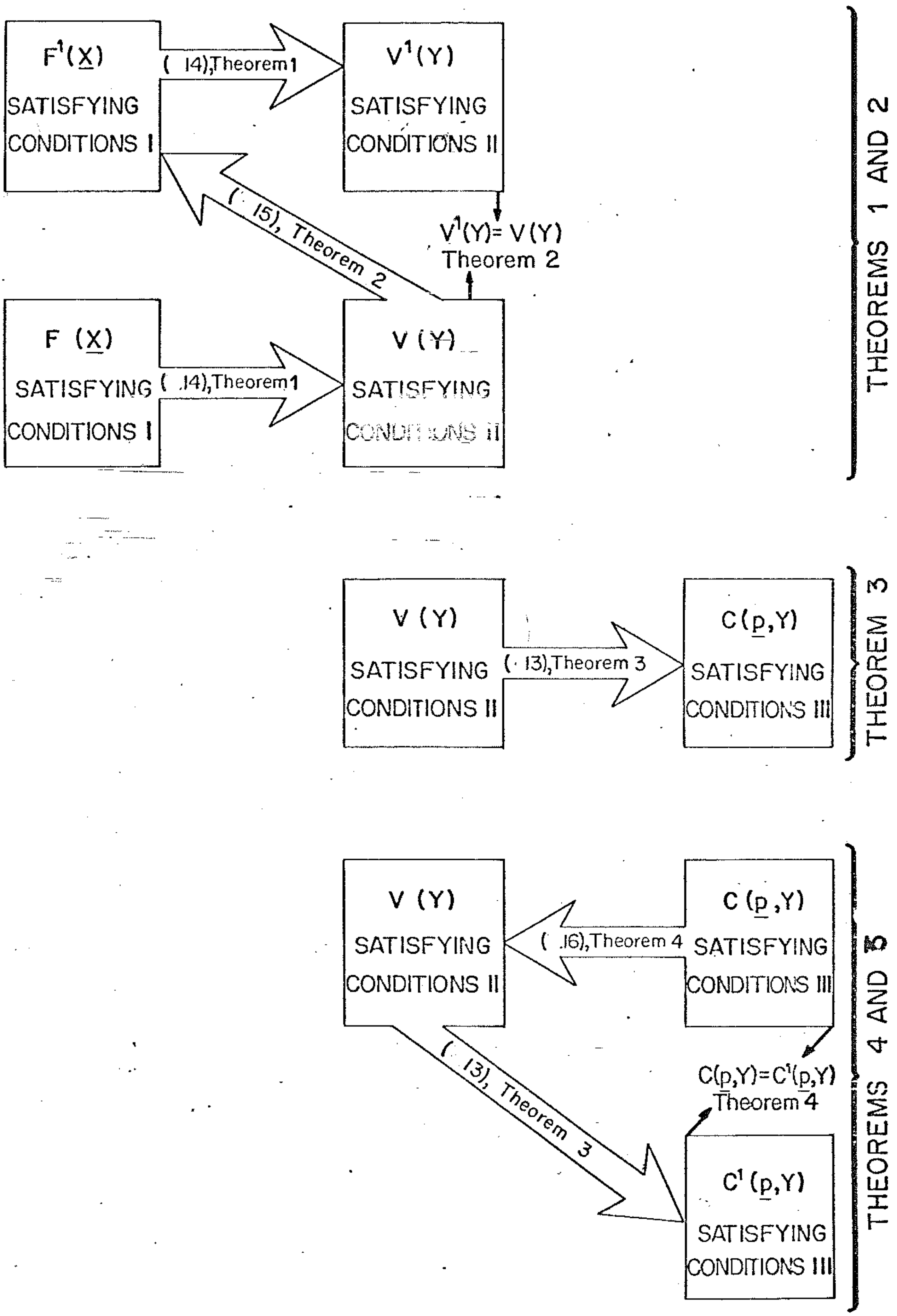


Figure: 4 SHEPHARD'S DUALITY THEOREM

The usefulness of the duality theory can now be stated. Suppose we are observing a world where there is only one technology that produces a certain product; in other words, there is only one production function $F(\underline{X})$ satisfying our required regularity conditions. By theorem I, we could derive a family of input requirement sets $V(Y)$, also satisfying its required regularity conditions. Associated with the technological requirements expressed in $F(\underline{X})$ and $V(Y)$ there is an observable cost function $C(\underline{p},Y)$ which also satisfy its regularity conditions by theorem 3.

It is possible to estimate either $F(\underline{X})$ or $C(\underline{p},\underline{Y})$ or both. The question is: if we want to analyse the factor demand functions or the factor share functions (as we actually do, studying technological change) where do we start? First let's observe that if we assume that there is only one technology, then Figure 4 collapses to Figure 5 where all equal $F(\underline{X})$, $V(Y)$ and $C(\underline{p},\underline{Y})$ were superimposed.(15)

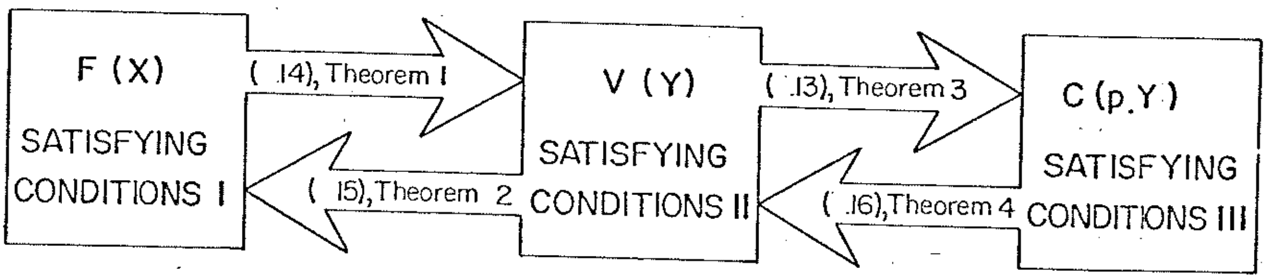


Figure. 5: SHEPHARD'S DUALITY THEOREM WITH A SINGLE TECHNOLOGY

(15) Observing Figure 4 it can be seen that if $F(\underline{X}) = F'(\underline{X})$ than it collapses to Figure 5. The proof can be found, <u>mutatis mutandis</u> in DIEWERT (1971), theorem 4.

Returning to our problem, it is possible to derive factor demand equations by observing $F(\underline{X})$ and go through the constrained optimization problem ( 8). Theorems 1 and 3 assure us that if $F(\underline{X})$ satisfies its regularity conditions, the derived cost function and therefore the demand functions, should be well-behaved. This route, although frequently pursued can be an arduous one, as we have noted.

Alternatively, it is possible to start by observing the cost functions, which presumably satisfy Conditions III, and using SHEPHARD's lemma arrive at the factor demand functions, certainly a much easier route to follow. The duality results represented in Figure 5 assure us that, associated with any cost function satisfying regularity conditions, there is an underlying technology which also satisfies regularity conditions. Thus, it is possible to recuperate all the information about technology without having to observe production functions themselves, since all economically relevant information about technology is embodied in the cost function. It suffices to estimate parameters of cost functions, and all technological information implied by production functions are known. This is the main statement of the duality theory in production analysis.[16]

An important aspect of SHEPHARD's theorem is that the regularity conditions are not really very strong. Maybe the

(16) We could generalize and say that any function which satisfies Conditions III is the cost function of a technology which satisfies regularity conditions.
Furthermore, two propositions proven by VARIAN (1978) on pages 145 and 42 are worth mentioning: (assume regularity conditions are satisfied)

a) If the technology exhibits constant returns to scale, the cost function $C(\underline{p},Y)$ is linear, homogeneous in Y, and may be written as $YC(\underline{p})$.
b) If the cost function is linear, homogeneous in Y, then the technology must exhibit constant returns to scale.

strongest assumption is the convexity of the input requirement sets V(Y) or the quasi-concavity of production functions F. Actually this requirement can be somehow relaxed, as UZAWA (1964) and VARIAN (1978) have shown.

By means of duality mapping described in Figures 1, 2 and 3, it is possible to construct a family of input requirements set $\tilde{V}(Y)$ which contains the underlying technology V(Y). If all regularity conditions are satisfied by the underlying technology V(Y) theorems 3 and 4 make $V(Y) = \tilde{V}(Y)$.

Suppose now that the original technology V(Y), from which the C(p,Y) used in the duality mapping was derived, is non-convex and non-monotonic. The constructed $\tilde{V}(Y)$ will be a convexified and monotonized version of V(Y) since we used in constructing $\tilde{V}(Y)$ a cost function which contained only the economically relevant information of the original technology. Figure 6 below illustrates this case.

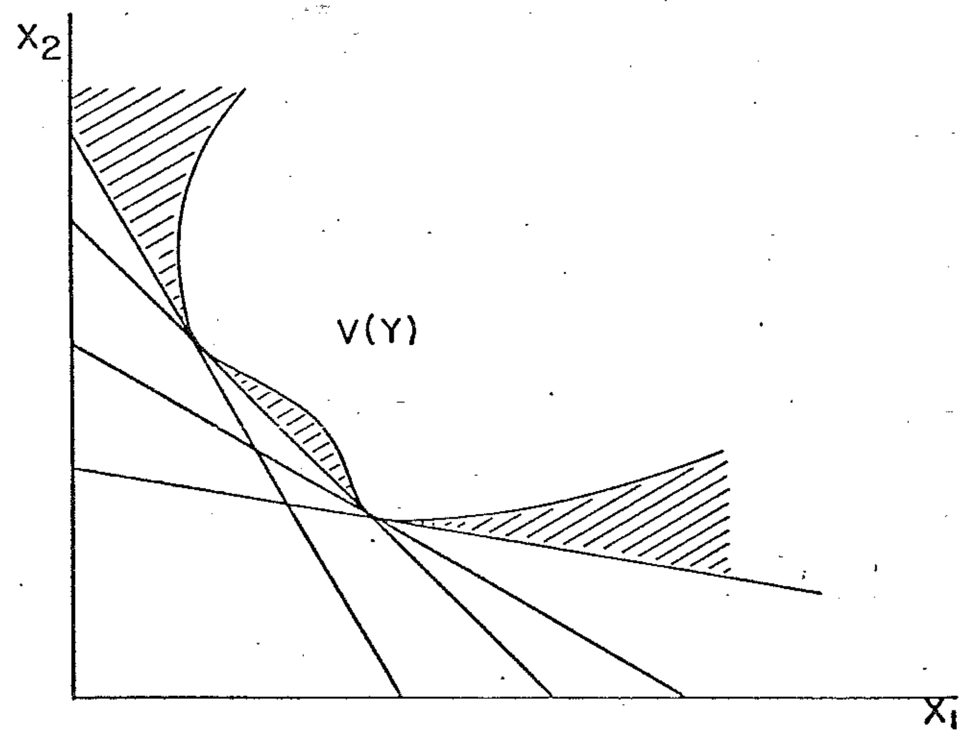


Figure 6 RELATION BETWEEN V(Y) AND THE CONVEXIFIED AND MONOTONIZED $\tilde{V}(Y)$

The constructed technology $\tilde{V}(Y)$ contains not only $V(Y)$ but also the cross-hatched areas, thus convexifying and monotonizing the original technology. Furthermore, the cost function $\tilde{C}(\underline{p},Y)$ generated by $\tilde{V}(Y)$ coincides with $C(\underline{p},Y)$, generated by $V(Y)$. In this sense, the assumption of convex input requirements sets can be relaxed while still generating meaningful cost functions.

## The Translog Cost Function

The translog cost function, *mutatis mutandis*, is defined in a strictly analogous way to the translog production functions. Then

$$C(\underline{p},Y) = H(Y)\; C(\underline{p}) = H(Y) \left[ \beta_o + \sum_i \beta_i \, \ell n p_i + \frac{1}{2} \sum_i \sum_i \beta_{ij} \, \ell n p_i \, \ell n p_j \right] \qquad (17)$$

where H is an increasing monotonic function with $H(0) = 0$, and represents the returns to scale specification of the function; for the function ( 17) to satisfy Condition III on Cost Functions for all $\underline{p} \gg 0$ it would be required that $\beta_i \geqslant 0$ for all i, $\sum_i \beta_i = 1$ and $\beta_{ij} = 0$ for all i,j, in which case ( 17) collapses to the COBB-DOUGLAS form.[17] If any $\beta_{ij} \neq 0$, then there is no simple set of parametric restrictions which assure us that regularity conditions are satisfied and therefore, like the translog production functions, they would have to be checked locally. Additional comments made about the translog production function ( 6) apply, *mutatis mutandis*, to the translog cost function ( 17).

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(17) The COBB-DOUGLAS is a self-dual form so that the dual and the primal belong to the same class of functions.

It is important to point out that in ( 17) have assumed that the cost function satisfies homotheticity conditions in which case the more general form $C(\underline{p},Y)$ is expressed as $H(Y)\ C(\underline{p})$ where $C(\underline{p})$ is the translog unit cost function.(18) The reason is that observed shifts in input combinations over an isoquant map are the result of three different effects: substitution effects along an isoquant, shifts of the isoquant caused by scale effects and technical change effects.

In classifying technological change

it is necessary to isolate effects caused by changes in factor intensity. Figure 7 shows unit isoquants in a two-factor world. Suppose that we do not know what the isoquants look like, and we want to explain an observed input combination change from point A at time t to point B at time t + 1. Both combinations produce one unit of output and are cost minimizing input bundles. The slope of ray OA defines the factor intensity at time t, our initial period of observation. Assuming that the production function is well-behaved, points A and B must be on different isoquants, called isoquants t and t + 1 respectively. The question is, what can we infer about the nature of the unit isoquant shift?

Albuquerque (1985) showed that a change in relative factor shares depends on the bias of technological change, on the elasticity of substitution and on changes in factor intensity.(18a) Furthermore, HICKS biased technological change was defined in terms of changes in relative factor shares at a

(18) Were homotheticity not assumed, the translog form would be applied to the more general form $C(p,Y)$, with $n + 1$ variables ($p_i$ for $i = 1 \ldots n$, and $Y$) instead of the $n$ variables of $C(p)$; $Y$ would be symmetrically introduced, as any other variable $p_i$.

(18a) $\hat{S} = \dfrac{\widehat{Sk}}{\widehat{SL}} = B + 1 - \dfrac{1}{\sigma}\ \hat{R}$ where $\hat{S}$ is the percentage change in share of capital ($Sk$) over share of labour ($SL$), $B$ is the degree of Hicks bias of technological change, $\sigma$ is the Allen elasticity of substitution and $R$ is the capital labour ratio.

given factor intensity level. Thus, in terms of Figure 7 we must check the effect of the isoquant shift on relative factor shares <u>along ray OA</u>. By doing this, the change in the factor shares ratio $\hat{S} = B$ and the bias of technological change is defined by observing if $\hat{S}$, evaluated at a constant factor intensity level is positive, negative or equal to zero, in which cases technological change would be capital-using, labour-using or neutral, respectively. The question now resumes to evaluating the change in relative shares at the point of the isoquant going through point B (isoquant t + 1) where it intersects ray OA, i.e., at poin[illegible]

At the cost-minimizing input bundle A, the relative factor price ratio equals minus the slope of isocost $C_0$. If the slope of the unit isoquant going throuhg B evaluated at where it intersects ray OA is steeper than the slope of isocost $C_0$, then technological change has been capital-using, if flatter, labour-using and if they are equal, technological change was neutral. As we have seen, relative factor prices, factor intensity and relative shares are not all independent. Thus an increase in the price of labour relative to that of capital, given a fixed factor intensity, implies an increase in the share of labour relative to that of capital. Since this is an aggregate analysis, whereby factor intensity is an exogenous variable dependent on relative factor supplies, the factor price changes necessary to induce factor use changes from input bundle A to input bundle B will be determined by its elasticity of substitution along the isoquant t + 1.

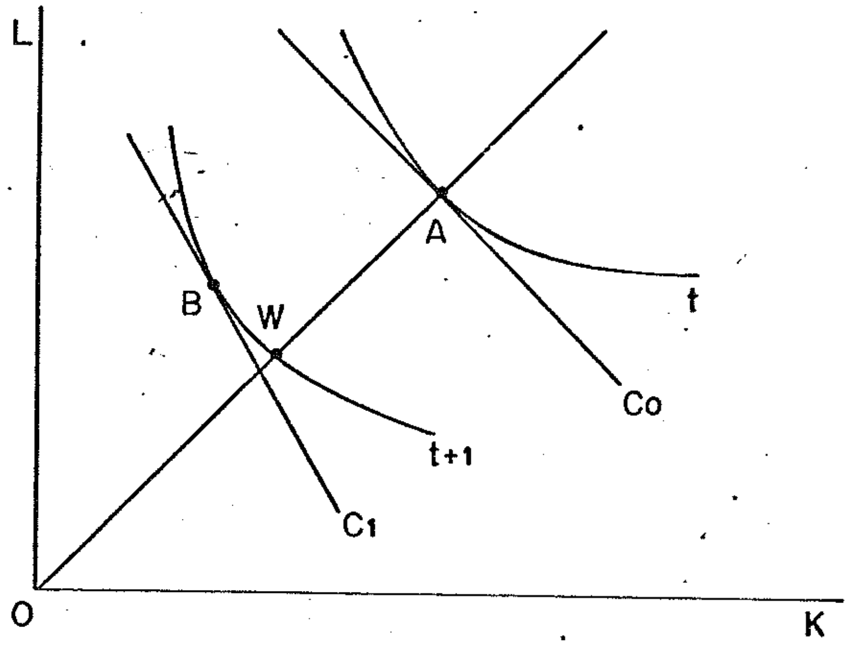


Figure 7 TECHNOLOGICAL CHANGE, SUBSTITUTABILITY AND SCALE EFFECT IN A UNIT ISOQUANT MAP

Suppose now, that in addition to technological change, isoquants also shift because of scale effects. So, the movement of the unit isoquant from t to t + 1 cannot be attributable to technological change only. In general, it would not be possible to identify technological biases, since it would not be possible to identify the shift in the isoquant due to technological progress alone. Measuring the slope of the isoquant t + 1 at W in Figure 7 and concluding that a certain change in relative factor share implies a certain type of bias, could not be correct. Actually, we would not be measuring technological effects alone but a combination of technological and scale effects on factor shares.

Of course, the problem could be avoided by postulating homogeneity of the production function. A weaker assumption however, that of homotheticity, would suffice. Homotheticity implies that scale effects shift the isoquants in a parallel way, maintaining factor shares constant with respect to scale.

Thus, any changes in factor shares measured along the ray OA can be attributed only to biased technological change.

BURGESS (1975) warns against pitfalls in the specification of technologies through the duality theory. He shows that estimates of elasticity of substitution based on the same set of data, differ substantially when estimated by a translog production function as compared to estimates by a translog cost function. The difference persists even assuming both translog functions to be approximations to a true function, rather than exact representations of a technology.

This is to be expected, since, unlike the COBB-DOUGLAS and the C.E.S. forms, the translog form is not self dual. If production and cost functions belong to the same class of functions, then it should not matter which one is actually used in econometric estimates (since they incorporate the same maintained hypotheses). But in the case of non-self-dual forms such as the translog, there is a difference involved, especially if they are considered exact representations of technology.[19] Of course, they could be considered as two different approximations to a true technology but the question then is, which one should be chosen? Since both forms can adopt the same behavioural assumption of cost minimization, we are left with no *a priori* reason for choosing one rather than another, and this should make interpretations of translog estimates less definitive and more conjectural.

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(19) In this case they are really representing two different technologies and there is not really much point in comparing them, unless such comparisons are based on goodness of fit which each one provides to the data. CHRISTENSEN et al (1973) actually pointed out this issue in their introduction to the translog form.

The translog cost function has been widely used, among others by BERNDT et al (1975), who estimated elasticities of substitution and tested for separability in a model incorporating energy as an explicit input in production. HALVORSEN (1977) also estimated energy substitution in US manufacturing, disaggregating energy inputs into electric energy, fuel oil, natural gas, and coal. HUMPHREY et al (1975) analysed substitutability between primary factors of production (capital and labour) and an aggregate of natural rescurce inputs, utilizing both the translog production function and the translog cost function as estimating equations. CHRISTENSEN et al (1976) analysed economies of scale in the US electric power-generating industry, by specifying initially a non-homothetic translog cost function and sequentially introduced restrictions of homotheticity, homogeneity and unitary elasticities of substitution; they used cross section data in an effort to avoid identification problems between scale effects and technical change.

Returning to the analysis of cost function in its most general formulation, it can be represented by

$$C = C(Y,\underline{p},t) \qquad (18)$$

where total cost is a function of the level of output Y, the vector of factor prices $p_i$ and time, standing for technological change. The translog representation of ( 18) can be written as

$$\ell nC = \beta_o + \sum_i \beta_i\ \ell np_i + \frac{1}{2}\sum_{ij}\sum \beta_{ij}\ \ell np_i\ \ell np_j + \beta_y\ \ell nY + \frac{1}{2}\beta_{YY}\ (\ell nY)^2 +$$

$$+ \beta_{yt}\ \ell nY(t) + \beta_t(t) + \frac{1}{2}\beta_{tt}(t)^2 + \sum_i \beta_{it}\ \ell np_i\ (t) +$$

$$+ \sum_i \beta_{iY}\ \ell np_i\ \ell nY \qquad i,j, = 1\ldots n \qquad (19)$$

with symmetry conditions $\beta_{vr} = \beta_{rv}$ [20] assumed as maintained hypotheses $(v,r = 1 \ldots n,t,y)$.

Equation ( 19) could be differentiated with respect to its arguments. Differentiating with respect to input price $p_i$, we have

$$\frac{\partial \ell nC}{\partial \ell np_i} = \frac{\partial C}{\partial p_i} \quad \frac{p_i}{C} \;;$$

but $\frac{\partial C}{\partial p_i} = X_i$ by SHEPHARD's Lemma, where $X_i$ is the derived demand for input i. Then

$$\frac{\partial \ell nC}{\partial \ell np_i} = \frac{X_i p_i}{C} = D_i$$

is the cost share attributable to input $X_i$. Thus,

$$D_i = \beta_i + \sum_i \beta_{ij} \; \ell np_j + \beta_{it}(t) + \beta_{iy} \ell nY \qquad (20)$$

$$i,j = 1 \ldots n$$

is a system of n equations showing cost shares as functions of input prices, technological change and scale of operation. Furthermore,

$$\frac{\partial \ell nC}{\partial (t)} = \beta_t + \beta_{tt}(t) + \beta_{yt} \; \ell nY + \sum_i \beta_{it} \; \ell np_i \qquad (21)$$

is the rate of technological progress, i.e., the reduction in cost due to the passage of time only, all other variables such as prices and quantities remaining constant. Last,

$$\frac{\partial \ell nC}{\partial \ell nY} = \beta_y + \beta_{yy} \; \ell nY + \beta_{yt}(t) + \sum_i \beta_{iy} \; \ell np_i \qquad (22)$$

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(20) This condition can be tested for, as a test of the translog specification itself, if equation ( 19) is estimated without the restriction imposed. See BURGESS (1974).

shows the effect on cost of changes in the level of output. The translog cost function can be estimated either directly through linear equation (19) or indirectly through systems composed of equations (19) and (20).[21]

Parametric restrictions can be imposed on equation (19) to make it satisfy some desired conditions.[22] Initially, one would like any cost function to satisfy Conditions III on Cost Functions.

Monotonicity and concavity are not satisfied globally by the translog cost function, except in the case where it collapses to the COBB-DOUGLAS form requiring

$$\beta_i > 0; \quad \sum_i \beta_i = 1; \quad \beta_{ij} = \beta_{it} = \beta_{iy} = 0 \qquad (23)$$ [23]

Locally however, the translog cost function can be checked for monotonicity in prices and in output. The conditions to be satisfied require that the set of estimated parameters are such that equations (20) and (22) are positive. The concavity in input prices condition requires that

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(21) Estimation of the unrestricted equation (19) would involve $1 + (n + 2) + \frac{(n + 2)\ (n + 3)}{2}$ parameters, while estimation of the differentiated translog cost function would reduce the number of parameters by 1, i.e., the constant terms in equation (19).

(22) Although the parametric restrictions are stated here in terms of the translog cost function, the same can be done, <u>mutatis mutandis</u>, to any other translog function, such as profit functions, revenue functions, production functions or utility functions.

(23) The usual homothetic COBB-DOUGLAS cost function would require that the following conditions be satisfied

$$\beta_i > 0; \quad \sum_i \beta_i = 1; \quad \beta_{ij} = \beta_{it} = \beta_{iy} = \beta_{yt} = 0 \qquad (24)$$

so that it could be expressed in the form

$$C = \left[ \bar{a} \prod_i p_i^{\beta_i} \right] H(y)\ A(t)$$

where H(Y) is positive monotonic, H(0) = 0 and A(t) is a term for HICKS neutral technology change. The translog cost function (and production function) can also be reduced to the C.E.S. form by the imposition of non-linear restrictions, requiring non-linear estimating methods.

a) the bordered HESSIAN matrix of $\frac{\partial^2 C}{\partial p_i \partial p_j}$ be negative definite

b) the HESSIAN matrix be negative semidefinite

c) the matrix $\frac{\partial^2 C}{\partial p_i \partial p_j}$ be negative semidefinite

within the range of input prices. BINSWANGER (1974b) has shown that the concavity condition can be translated into the condition that the matrix $\left[ \sigma_{ij} \right]$ of partial elasticities of substitution be negative semidefinite.

Linear homogeneity in prices is a condition which must be satisfied by a cost function. Given our translog cost function $\ell n\, C(Y,\underline{p},t)$, linear homogeneity in prices is satisfied if $\ell n\, C(Y,\lambda \underline{p},t) = \ell n\, C(Y,\underline{p},t) + \ell n\, \lambda$. The following restrictions on equation (3.19) would assure that this condition is met:

$$\sum_i \beta_i = 1; \ \sum_i \beta_{it} = \sum_i \beta_{iy} = \sum_i \beta_{ij} = \sum_j \beta_{ij} = \sum_i \sum_j \beta_{ij} = 0^{(24)} \qquad (25)$$

Other important restrictions can be imposed on the translog cost function. They are not essential features of regular cost functions, but are important conditions which the translog form allows us to test for, and which have been often assumed as maintained hypotheses in traditional econometric models based on COBB-DOUGLAS or C.E.S. functions. They are homotheticity and homogeneity.

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(24) Linear homogeneity in prices implies (n + 3) restrictions in the general translog cost function. The last two equalities of ( 25) are not independent restrictions, since it is implied by the equality stated before it and by the symmetry assumption.

Homotheticity of production in inputs is a desirable assumption of cost functions if we want to analyse technological change. As stated above, it means that isoquants shift in a parallel way with respect to scale effects leaving distributive shares undisturbed. Homotheticity implies that the cost function can be written as a separable function in output on the one hand, and factor prices and time on the other. Thus, $C(Y,\underline{p},t) = C^*(\underline{p},t)\ H(Y)$. The following restrictions imply that the cost function (3.19) is homothetic:

$$\beta_{yt} = \beta_{iY} = 0 \qquad i = 1 \ldots n \tag{26}$$

A stronger assumption is that of homogeneity, implying constant returns to scale (linear homogeneity) or a constant degree of returns to scale (homogeneity of degree s). Homogeneity implies, in addition to restriction ( 26), that the elasticity of cost with respect to output is constant (= s ). Thus, it implies that $\ell n\left[C^*(\underline{p},t)\ H(\lambda Y)\right] = \ell n\left[C^*(\underline{p},t)\ H(Y)\right] + s\ \ell n\lambda$ which can be satisfied if

$$\beta_Y = s;\ \beta_{YY} = 0 \tag{27}$$

and constant returns to scale implies $s = 1$.(25)

Traditionally, econometric studies have ignored technological change by assuming a fixed technology. While this assumption may be acceptable for short-run analysis, it certainly lacks realism when applied to long-run studies. Together with the assumption of constant returns to scale (or

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(25) Homotheticity implies (n + 1) restrictions on ( 19); homogeneity implies one additional restriction; constant returns to scale, for instance, implies (n + 3) restriction on the general translog cost function, since $\beta_Y = 1$ is an additional constraint.

the weaker assumption of homotheticity) it means that factor shares remain constant but for changes in the exogenously given factor proportion ratio (in the case of aggregate analysis) or in the exogenously determined factor price ratio (in the case of disaggregated analysis). This can be seen by recalling that under these assumptions factor shares changes are determined solely by movements along an isoquant, assuming no technological change, i.e.,

$$\hat{S} = 1 - \frac{1}{\hat{R}}$$

Alternatively, the issue of technological change has been avoided, while its existence, is still acknowledged by assuming a slightly weaker assumption than no technological progress, namely HICKS neutrality.(26) By the same reasons, changes in factor input mix and factor shares would not come about unless exogenously imposed or through changes in market determined input prices.

UZAWA (1964) has shown that a HICKS neutral technological change can be written as $F(\underline{X},t) = J(t)\ F(\underline{X})$ where $J(t)$ is a function of time. In terms of its dual cost function, it implies that $C(Y,\underline{p},t) = J^*(t)\ C(Y,\underline{p})$. Thus, HICKS neutrality can be imposed on equation (3.19) by making it satisfy the following restrictions.(27)

$$\beta_{yt} = 0; \quad \beta_{it} = 0 \qquad i = 1 \ldots n \qquad (\ 28)$$

The general translog cost function ( .19) could have been written as $\ell nC = C(\ell n\underline{p}, \ell nY, f(t))$ instead of assuming, as in ( .19), that $f(t) = t$. WILLS (1979) estimated translog cost functions using various alternative specifications. One of them was a biased technical change model assuming $f(t) = t$.

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(26) See, for instance, BERNDT et al (1973a). Their approach, described above, was followed by a number of authors using the translog function framework.

(27) Imposing HICKS neutrality implies (n + 1) restrictions in equation ( 19).

TODA (1974), using a translog production function, made $f(t) = t^2$. Both models introduced technological change by making it a function of the index "t" which, due to the dynamic nature of technical process, represents a continuous process of changes in technology. In this framework, not assuming HICKS neutrality, allows for measurement of bias of technological change, i.e., the $\beta_{it}$ terms are estimates of factor-using (or saving) biases since they measure the change in factor shares due to the passage of time only, keeping all other variables constant. From equations (20) and recalling the meaning of equations (21) (22) and (23) we have a measure of bias as (assuming $f(t) = t$)

$$\frac{\partial D_i}{\partial t} = \beta_{it} \qquad i = 1 \ldots n \qquad (29)$$

This model also allows for a simultaneous neutral component of technological change. As implied by neutrality, the measure of the neutral component is not in equations (20) since neutrality does not affect factor shares. But it is direcly measurable from equation (21) and equals $\left[\beta_t + \beta_{tt}(t)\right]$. The remaining terms in equation (21) show the effects on the rate of technological change of the biased components $(\sum_i \beta_{it}\ \ell n p_i)$ and a term $(\beta_{yt}\ \ell nY)$ which captures the effects of changes in scale as if they were due to changes in technology. Recall that by assuming homotheticity, conditions (26) have to be imposed, eliminating this last term from equation (21) and also the term $(\beta_{iy}\ \ell nY)$ from equations (20), another source of indeterminacy between effects of technological changes and scale changes.[28]

## The Factor-Augmenting Translog Cost Function

Alternatively, technological change may be introduced by replacing the index t (or b(t)) by an index of productivity A(t). Thus, the translog cost function could be written as $C(\ell nY, \ell n\underline{p}, \ell nA(t))$. This general form, where all t's were replaced by $\ell nA(t)$, yields the same interpretation with respect to the parameters β's, the only difference being the more realistic assumption of not necessarily assuming that technological change occurs smoothly over time as the index t suggests.

If we assume that all productivity changes are factor-augmenting, then $C(\ell nY, \ell n\underline{p}, \ell nA(t))$ can be written as $C(\ell nY, \ell n\underline{p} + \ell nA(t))$. The following (n + 3) restrictions would apply in ( 19)[29]

$$\begin{aligned}
\beta_{yt} &= 0 \\
\beta_{t} &= \sum_i \beta_i \frac{\partial \ell nA_i}{\partial t} \\
\beta_{tt} &= \sum_i\sum_j \beta_{ij} \frac{\partial \ell nA_i}{\partial t} \frac{\partial \ell nA_j}{\partial t} \qquad i = 1 \dots n \qquad (30) \\
\beta_{it} &= \sum_j \beta_{ij} \frac{\partial \ell nA_j}{\partial t}
\end{aligned}$$

---

(28) Homotheticity must be assumed if technological progress is identified to changes in measured, or observed, total cost. Of course, the coeteris paribus condition must be satisfied (i.e. constant input ratios), a highly unlikely event, or else, cost data must be adjusted to reflect constant conditions of production. The estimation of the translog cost curve allows technological change to be measured without assuming homotheticity, a solution to the so-called "impossibility" problem in measuring technological progress.

(29) Of course, all t's would have to be replaced in translog cost function by appropriate $\ell nA_i$ (t) and the $\ell np_i$ would become $\ell n\,(p_iA_i)$.

so that the translog cost function with factor-augmenting would, become

$$\ell nC = \beta_0 + \sum_i \beta_i \ell np_i + \frac{1}{2} \sum_{ij}\sum \beta_{ij} \ \ell np_i \ \ell np_j +$$

$$+ \beta_y \ell nY + \frac{1}{2} \beta_{yy} \ \ell nY^2 + \sum_i \beta_{iy} \ \ell np_i \ \ell nY +$$

$$+ \sum_i \beta_i \ \ell nA_i + \frac{1}{2} \sum_{ij}\sum \beta_{ij} \ \ell nA_i \ \ell nA_j + \sum_{ij}\sum \beta_{ij} \ \ell np_i \ \ell nA_j +$$

$$+ \sum \beta_{iy} \ \ell nA_i \ \ell nY \qquad (31)$$

Equations (20), (21) and (22) become

$$D_i = \beta_i + \sum_j \beta_{ij} \ \ell np_j + \beta_{iy} \ \ell nY + \sum_j \beta_{ij} \ \ell nA_j \qquad (32)$$

$$\frac{\partial \ell nC}{\partial t} = \sum_i \left( \frac{\partial \ell nC}{\partial \ell nA_i} \ \frac{\partial \ell nA_i}{\partial t} \right) = \sum_i \left[ \beta_i + \sum_j \beta_{ij} \ell nA_j + \sum_j \beta_{ij} \ell np_j + \right.$$

$$\left. + \beta_{iy} \ \ell nY \right] \frac{\partial \ell nA_i}{\partial t} \qquad (33)$$

$$\frac{\partial \ell nC}{\partial \ell nY} = \beta_y + \beta_{yy} \ell nY + \beta_{iy} \ (\ell np_i + \ell nA_i) \qquad (34)$$

HICKS neutrality in factor-augmenting form requires that $C(\ell n\underline{p} + \ell n\underline{A}, \ \ell nY) = H(\ell n\underline{A}) \ C^*(\ell n\underline{p}, \ \ell nY)$. The terms involving $\ell nA_i$ are

$$\sum_i \beta_i \ \ell nA_i + \frac{1}{2} \sum_{ij}\sum \beta_{ij} \ \ell nA_i \ \ell nA_j + \sum_{ij}\sum \beta_{ij} \ \ell np_i \ \ell nA_j +$$

$$+ \sum_i \beta_{iy} \ \ell nA_i \ \ell nY \qquad (35)$$

This expression can be a function of the fact- augmentation coefficients alone, only if $\beta_{iy} = 0$ and $\beta_{ij} = 0$, which reduces our cost function to the trivial COBB-DOUGLAS case.(30)

If we assume equal rates of factor augmentation and homogeneity in prices of cost functions, $A_i(t) = A_j(t)$, then (3.35) becomes

$$\ell nA\ (1 + \sum_{ij}\sum \beta_{ij}\ \ell np_i) \qquad (\ 36)$$

Now, ( 36) cannot be equal to zero unless $\beta_{ij} = 0$, in which case we return to the COBB-DOUGLAS function. Thus, we do not know how to express HICKS neutrality parametrically in an unrestricted factor augmentation framework.(31)

Assume now that the <u>rate of factor augmentation is constant</u>, i.e. the $A_i(t)$ are such that the price variables in the translog cost functions are "corrected" by the increased efficiency of the input i. Thus the price of factor i in efficiency units is

$$p_i^* = p_i\ \exp\ (a_i t) \qquad (\ 37)$$

such that $C(\ell nY, \ell n\underline{p} + \ell n\underline{A}(t)) = C(\ell nY, \ell np_i^*)$ and $A_i(t) = \exp\ (a_i t)$. Substituting into equation ( 31)

$$\ell nC = \beta_0 + \sum_i \beta_i\ \ell np_i + \frac{1}{2} \sum_{ij}\sum \beta_{ij}\ \ell np_i\ \ell np_j + \beta_y\ \ell nY + \frac{1}{2}\beta_{yy}\ \ell nY^2 +$$
$$+ \sum_i \beta_{iy}\ \ell np_i\ \ell nY + \sum_i \beta_i\ a_i t + \frac{1}{2}\sum_{ij}\sum \beta_{ij}\ a_i a_j\ t^2 +$$
$$+ \sum_{ij}\sum \beta_{ij}\ a_j\ \ell np_i\ t + \sum_i \beta_{iy}\ a_i t\ \ell nY. \qquad (\ 38)$$

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(30) $\beta_{iy} = 0$ together with the factor augmentation restriction $\beta_{yt} = 0$ means that the function has to be homothetic. The remaining restrictions $\beta_{ij} = 0$, reduce the translog cost function to the trivial COBB-DOUGLAS case (assuming homogeneity in prices).

(31) We can still test for HICKS neutrality, i.e. test for $\beta_{ij} = 0$ and $\beta_{iy} = 0$; of course, such a test would really be a test for the COBB-DOUGLAS framework.

Equation ( 38), compared to equation ( 19), has n new parameters, i.e. the constant rates of factor augmentations $a_i$'s. (32)

WILLS (1979) used the factor-augmenting framework to estimate rates and biases of technological change, assuming constant rates of factor augmentation.

BINSWANGER (1973, 1974b) estimated rates and biases of technological progress using a somewhat different technique which allowed for relaxation of the assumption requiring constant rates of factor augmentation.(33) His method involved the independent estimation of the $\left[\beta_{ij}\right]$ matrix using cross-section data assuming, therefore, no changes in technology. The estimated matrix of $\beta_{ij}$ coefficients is then used to generate a series of factor-augmenting coefficients. Assuming constant returns to scale and factor-augmenting technological change the cost share equations ( 20) can be written as

$$D_i = \beta_i + \sum_j \beta_{ij} \ \ell n \ N_j$$

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(32) Even assuming constant rates of factor augmentation, we collapse to the C.D. case in imposing HICKS neutrality parametrically. This is not the case, however, assuming equal and constant rates of factor augmentation, in which case the restrictions to be imposed are $\Sigma\beta_{iy} = 0$ and $\beta_{ij} = 0$, a slightly weaker restriction than the COBB-DOUGLAS case which forces all $\beta_{iy} = 0$. But then, it means that homotheticity is not assumed (it requires $\beta_{iy} = 0$), in which case scale effects are not isolated from technological change.

(33) He actually used two models. His "Model B", assuming constant rate of bias, is a model like our equation (3.19) where he introduced $f(t) = t$ and assumed homotheticity. Since he estimated the parameters of the cost function using its derivative with respect to factor prices (i.e. cost share equations) all the terms involving the level of output dropped out, leaving only those involving factor prices and time. His "Model B" did not assume factor-augmenting technological change. His "Model A", assuming factor augmentation, is described in the text.

where $N_j$ is the input price corrected for efficiency changes (i.e. $N_j = \frac{P_j}{A_j}$). Differentiating totally, we have

$$dD_i = \sum_j \beta_{ij}\, d\ell n\, N_j = \sum_j \beta_{ij}\, (d\ell np_j - d\ell n\, A_j) \qquad i = 1 \ldots n$$

Imposing the necessary restrictions due to the fact that $\Sigma D_i = 1$, and thus the matrix $\left[\beta_{ij}\right]$ is not of full rank, it is possible to generate a series of factor augmentation coefficients as follows:

$$dD_i = \sum_j^{n-1} \beta^*_{ij} \left[(d\ell np_j - d\ell np_n) - (d\ell nA_j - d\ell nA_n)\right]$$
$$i,j = 1,2\ldots(n-1)$$

where $\left[\beta^*_{ij}\right]$ is a square matrix of dimension $(n - 1)$, from where it follows that

$$\underline{da} = \underline{dp}^* - \left[\beta^*_{ij}\right]^{-1} \underline{dD}\ ,$$

where $\underline{da}$ is the vector $\left[d\ell nA_j - d\ell nA_n\right]$, $\underline{dp}^*$ is the vector $\left[d\ell np_j - d\ell np_n\right]$, and $\underline{dD}$ is the vector of observed factor shares.

As we can see this method of generating estimates of factor-augmenting coefficients does not require the assumption of constant rates of factor augmentation, but it does require the independent estimation of the matrix $\left[\beta_{ij}\right]$, ideally assuming that all production units are in the same production function.(34)

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(34) BINSWANGER (1973) used cross-section data to estimate $\left[\beta_{ij}\right]$, assuming that all observational units had equal technological possibilities available to them. Note that from $\underline{dD} = \left[\beta^*_{ij}\right] \underline{dp}^* - \left[\beta^*_{ij}\right] \underline{da}$ it is possible to estimate the changes in factor shares which would have occurred in the absence of any price changes, i.e., those changes in factor share due solely to technological change, $\underline{dD}^*$. To do so, set $\underline{dp}^* = 0$, which yields

$$\underline{dD}^* = - \left[\beta^*_{ij}\right] \underline{da} = \underline{dD} - \left[\beta^*_{ij}\right] \underline{dp}^*$$

It should be mentioned that it is possible to introduce technological change in the translog cost functions assuming both HICKS neutral and HICKS biased factor-augmenting technological change.[35] This would be possible writing $C(\ell nY, \ell n\underline{p}, t)$ as $J^*(t)\, C(\ell nY, \ell n\underline{p} + \ell n\underline{A})$ requiring, therefore, the interpretation of the factor augmentation coefficient $A_i(t)$ as an index of factor productivity change and $J^*(t)$ as technological change due to the passage of time per se. This [illegible] would require the imposition on equation (19) of restrictions (28) associated with HICKS neutrality as well as restrictions ([illegible]) associated with the factor-augmenting representation of technological change.

Last, it should be mentioned that issues of functional separability can be conveniently tackled by the translog function framework. BERNDT et al (1973b) have developed the set of parametric restrictions required for various types of separability in terms of homothetic translog production functions. The separability conditions on cost functions are analogous to those in production functions.

Let us assume that there exists a cost function $C(Y,\underline{p})$ characterized by homotheticity in production, such that it can be written as $C(Y,\underline{p}) = H(Y)\, C(\underline{p})$ where $H(Y)$ is an increasing monotonic function of output only, and $C(\underline{p})$ is the unit cost of production as a function of input prices only. Let us further assume that there is no technological change, or, alternatively, accept the weaker assumption of HICKS neutral technological progress. Then, separability in terms of a cost function can be defined in an exactly analogous way as the definition stated

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(35) Using JORGENSON et alii (1971) terminology, this involves introducing both output-augmenting and factor-augmenting technological change.

by equation ( 3), after replacing the production function F(.) by the unit cost function C(.). The qualifications of weak separability, strong separability and complete separability apply to cost functions in a strictly parallel way.[36]

It should be noted that the definition of the concept of separability involves the independence of the marginal rate of substitution between selected pairs of inputs, with respect to quantities of third inputs. If the existence of such independence is the criterion that discriminates separable from non-separable production technologies, then it becomes necessary to distinguish when a possible dependence of the marginal rates of substitution, with respect to quantities of third inputs, is a result of functional separability, and when it is a result of technological change or scale effects. By the same reason that, as explained in Figure 7, it becomes necessary to account for non-homotheticity if we are to identify biases in technological change, it becomes necessary to assume that possible non-parallel shifts of the unit isoquant caused by changes in quantities of third factors, cannot be caused by either heterothetic or HICKS biased shifts of the unit isoquant.

Unless such assumptions are made, it is not possible to classify a production process as to its separability characteristics, scale and technological effects, all simultaneously. Thus, restrictions ( 26) and ( 28) should be imposed on the translog cost function ( 19) becoming

$$\ell nC = \beta_0 + \sum_i \beta_i \ \ell np_i + \frac{1}{2} \sum_{ij}\sum \beta_{ij} \ \ell np_i \ \ell np_j + \\ + \beta_Y \ \ell nY + \frac{1}{2} \beta_{YY} \ (\ell nY)^2 + \beta_t (t) + \frac{1}{2} \beta_{tt} \ t^2 \qquad (39)$$

(36) See BERNDT et al (1973b) and LAU, in FUSS et al (1978).

The separability condition that must be satisfied for inputs i and j to be separable from input h is

$$C_j C_{ih} - C_i C_{jh} = 0 \qquad (3.40)$$

Weak separability requires (3.40) to hold for inputs i and j in one partition and h in another; strong separability requires (3.40) to hold for inputs i, j and h, all belonging to distinct partitions; complete separability requires, in addition to strong separability, that each partition possesses only one input. Differentiating (3.39) and substituting in (3.40) we get the separability condition

[illegible] $_{jh} - D_j \beta_{ih} = 0 \qquad (3.41)$

As shown by BERNDT et al (1973b), condition (3.41) is satisfied if

a) $\beta_{jh} = \beta_{ih} = 0$ , in which case the elasticities of substitution $\sigma_{jh} = \sigma_{ih} = 1$ , or

b) $\dfrac{\beta_i}{\beta_j} = \dfrac{\beta_{ih}}{\beta_{jh}} = \dfrac{\beta_{ir}}{\beta_{jr}}$ , in which case the elasticities of substitution $\sigma_{ih} = \sigma_{jh} \neq 1.$ (37)

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(37) In arriving at the non-linear restrictions (b) in their original work BERNDT et al (1973b), using a translog production function, required the assumption of constant returns to scale, so that, in terms of the production function parameters, the equivalent of the conditions (b) above, were $\dfrac{\alpha_i}{\alpha_j} = \dfrac{\alpha_{ih}}{\alpha_{jh}} = \dfrac{\alpha_{ir}}{\alpha_{jr}}$. Now, when using the translog cost function, the conditions

$$\sum_i \gamma_{ij} = \sum_j \gamma_{ij} = \sum\sum_{ij} \gamma_{ij} = 0$$

required to derive the non-linear separability conditions, are those necessary for linear homogeneity in prices, a condition which every cost function must satisfy. Thus, using the translog cost function it is not necessary to require constant returns to scale as a maintained hypothesis, a clear advantage of using the dual instead of the primal.

## BIBLIOGRAPHY

ALBUQUERQUE, M.C.C. de - A Translog Analysis of Technological Change and Scale Effects in Brazilian Agriculture - A case of Inefficient Modernization. Pg.D. Thesis, Harvard University, University Microfilm International, Ann Arbor, MI, 1985.

— The Economic Theory of Technological Change — EAESP/FGV — mimeo 1985

ALLEN, R.G.D. - Mathematical Analysis for Economics. MacMillan and Co., St Martin's Street, London, 1938.

BERNDT, E.R. and CHRISTENSEN, L.R. - The Translog Function and the Substitution of Equipment, Structures and Labor in Us Manufacturing - 1929-1968. Journal of Econometrics, 1, 1973 (a).

- The Internal Structure of Functional Relationship: Separability, Substitution and Aggregation. Review of Economic Studies, July, 1973 (b).

BERNDT, E.R. and WOOD, D.O. - Technology, Prices and the Derived Demand for Energy, Review of Economics and Statistics, August, 1975.

BINSWANGER, H.P. - The Measurement of Biased Efficiency Gains in United States and Japanese Agriculture to Test the Induced Innovation Hypothesis. Ph.D. Thesis. North Carolina

State. University at Raleigh. University Microfilms 73-29, 528, 1973.

- The Measurement of Technical Change Biases with Many Factors of Production. American Economic Review, V. LXIV, nº 4, September, 1974.

- A Cost Function Approach to the Measurement of Elasticities of Factor Demand and Elasticities of Substitution. American Journal of Agricultural Economics, May, 1974(a).

- A Microeconomic Approach to Induced Innovation. Economic Journal, V. 84, December, 1974 (b)

BURGESS, D.F. - Production Theory and the Derived Demand for Imports. Journal of International Economics 4, 1974.

- Duality Theory and Pitfalls in the Specification of Technologies. Journal of Econometrics 3, 1975.

CHRISTENSEN, L.R - Concepts and Measurement of Agricultural Productivity. American Journal of Agricultural Economics, December, 1975.

CHRISTENSEN, L.R. and GREENE, W.H. - Economies of Scale in US Eletric Power Generation. Journal of Political Economy, 84, 1976.

CHRISTENSEN, L.R., JORGENSON, D.W. and LAU, L.J. - Conjugate Duality and the Transcendental Logarithmic Function. Econometrica, 39, July (abastract), 1971.

- Trasncendental Logarithmic Production Frontiers. Review of Economics and Statistics, February, 1973.

CORBO, V. and MELLER, P. - The Translog Procution Function: Some Evidence from Establishment Data. Journal of Econometrics, 10, 1979.

DENNY, M. and MAY, D. - The Existence of a Real Value-Added Function in the Canadian Manufacturing Sector. Journal of Econometrics, 5, 1977.

DIEWERT, W.E. - An Application of the Shepard Duality Theorem: A Generalized Leontief Production Function. Journal of Political Economy, May/June, 1971.

- Applications of Duality Theory, in INTRILLIGATOR M.D. and KENDRICK, D.A. (eds), Frontiers of Quantitative Economics, V. 2. North Holland Publishing Company, N. York, 1974.

FUSS, M. and MACFADDEN, D. (eds) - Production Economics: A Dual Approach to Theory and Applications. Contributions to Economic Analysis, V. 1. North Holland, New York, 1978.

FUSS, M., MCFADDEN, D. and MUNDLAK, Y. - A Survey of Functional Forms in the Economic Analysis of Production, in FUSS et al (1978).

GOLDMAN, S.M. and UZAWA, H. - A Note on Separability in Demand Analysis. Econometrics, V. 32(3), 1964.

HALVORSERN, R. Energy Substitution in US Manufacturing. Review of Economics and Statistics, November, 1977.

HUMPHREY, D.B. and MORONEY, J.R. - Substitution Among Capital, Labor and National Resource Products in American Manufacturing. Journal of Political Economy 83, 1975.

JORGENSON, D.W., CHRISTENSEN, L.R. and LAU, L.J. - Conjugate Duality and the Transcendental Logatithmic Function. Harvard University (mimeo); alsi in Econometrica, July, 1971.

MCFADDEN, D. - Constant Elasticity of Substitution Production Functions. Review of Economic Studies, June, 1963.

- Cost, Revenue and Profit Functions: A Cursory Review. Working Paper 86, IBER., University of California, Berkeley, March, 1966.

MUKERJI, V. - A Generalized SMAC Function with Constant Ratios of Elasticity of Substitution. Review Economic Studies, Oct, 1963.

RUSSEL, R.R. - Functional Separability and Partial Elasticities of Substitution, Review of Economic Studies. V. 42(I), 1975.

SHEPARD, R.W. - Cost and Production Functions, N.J. Princeton University Press, 1953.

TODA, Y. - Capital-Labour Substitution in Production Functions: The Case for Soviet Manufacturing for 1950-71 in ALTAMAN et alii (1974).

UZAWA, H. - Production Functions with Constant Elasticity of Substitutions, Review of Economic Studies, V. 29(4), 1962.

- Duality Principles in the Theory of Cost and Production. International Economic Review, V. 5, nº2, May, 1964.

VARIAN, H.R. - Microeconomic Analysis. W.W. Norton Co., New York, 1978.

WALTERS, A.A. - Production and Cost Functions: An Econometric Survey. Econometrica, Jan/april, 1963.

WILL, J. - Technical Change in the US Primary Metals Industry. Journal of Econometrics, nº 10, 1979.